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Homothermal acceleration waves in nematic liquid crystals

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Abstract

The aim of this paper is the study of propagation of acceleration waves of arbitrary shape in nematic liquid crystals. The development of balance equation reduced to singular surface and the application of Hadamard's theorem permit to obtain the speeds and the conditions of propagation of the acceleration waves. Differential equations that describe the modifications of the metric and topological properties of the wave during the propagation are deduced in function of kinematical descriptors of the continuum and its thermodynamical state. The deduction of the coefficients of evolution equation for the amplitude of the jump concludes the paper. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In this paper, we study the propagation and the growth and decay of acceleration waves in nematic liquid crystals. An acceleration wave is a moving surface intersecting a body, across which the accelerations of the kinematical descriptors of the body suffer jumps of finite size.

Several authors have investigated the properties of acceleration waves for different materials. Chen (1968) studied the influences of the thermodynamic properties of simple materials on the propagation, growth and decay of acceleration waves. Bowen (1969) established the properties of plane acceleration and higher order waves propagating into mixtures of elastic materials without diffusion but with a non-zero chemical affinity. Chadwick and Currie (1972) restricted their studies to elastic heat conductors. Wright (1973), in an extremely detailed article, studied acceleration and higher order discontinuities waves in simply elastic materials, with particular attention to multiple and non-uniform velocities of propagation and to the formation of caustics. Nunziato and Walsh (1977) got some results for one-dimensional acceleration waves in granular materials while Lindsay and Straughan (1979) examined the evolutionary behaviour of acceleration waves in perfect fluids. Ottosen and Runesson (1991) have taken a spectral analysis for acceleration waves in elasto-plastic materials. Recently, Mariano and Sabatini (1999a,b) proposed a description of the propagation of acceleration waves in general continua with microstructure (multifield theories of solids).

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The mathematical model of continuum with microstructure allows a general description of a wide range of materials whose behaviour is influenced by its fine structure. Thus, the term microstructure is usually related to the material texture of the body or to additional structures brought about by the external environment or by mathematical schemes. Typical examples of the latter family may be found within the set of models describing the mechanical behaviour of one or two-dimensional structural elements such as beams, plates and shells. Such structural elements are studied by reducing the body motion to that of a representative line or surface and of a vector field on it accounting for the behaviour of the transversal sections (Antman, 1960, 1995; Naghdi, 1960). More refined models of beams with affine structure (tubes) consider second order tensor valued fields.

In general, within the setting of multifield theories, to each material point **P** two fields are assigned: the former represents the placement of **P** in the Euclidean space, while the latter (order parameter field) takes values on a finite dimensional manifold $\mathscr{M}$ and describes all possible configurations of the microstructure. In this way, the order parameter is considered as an observable quantity (in the sense that external observers should take two different measures to recognise the configuration of the body). So, interactions should be associated to the order parameter itself. If the manifold $\mathscr{M}$ is endowed by a physically significant connection it is possible to describe the interactions between the elements of the microstructure by microstresses and self-forces that satisfy appropriate balance equations (Capriz, 1985, 1989).

In this paper, the order parameter field describes a preferential orientation of slender molecules of nematic liquid crystals and it is identified with a vector field on the body.

A liquid crystal is a mesomorphic state of the matter. It has the characteristic fluidity of liquids and optical properties of solids. The centroids of molecules present an ordered structure like the solid lattice but the molecules are oriented randomly. If the temperature causes changes of phase in a liquid crystal, it is said to be thermotropic; conversely, it is said to be lyotropic if the change is given by a different concentration of solvent. In 1922, Friedel proposed a classification of liquid crystals in *nematics*, *cholesterics* and *smectics* with increasing complexity (Virga, 1994). The smectic phase is characterised by two-dimensional stratified structures with molecules arranged in layers. The molecules of a cholesteric liquid crystal have the form of helical springs with right-handed or left-handed wrapping (chirality): they have a notable symmetry for which it is not possible to distinguish the head from the tail, but a mirror symmetry changes the chirality of molecules. Finally, the term *nematic* comes from ancient Greek where it assumes the meaning of "thread"; the molecules of a nematic liquid crystal are rod-shaped with typical dimensions from 5 to 20 Å and have a complete mirror symmetry with respect to their mid-section. For a complete description of a nematic liquid crystal it is necessary to specify the motion of a particular point of a molecule (for example the centroid) and its change of orientation.

With reference to acceleration waves, we found that

- The directions of propagation of an acceleration wave are the eigenvectors of a particular second order tensor that generalises the acoustic tensor of elastic materials.
- The equation that describes the growth and the decay of the jump across the surface of the accelerations is of Bernoulli's type with non-constant coefficients.
- The growth and decay of the amplitude of the jump is influenced not only by the rheological properties of material but also by the response of material ahead of the wave.
- At every point, the configuration of the discontinuity surface depends on the macrostate, the microstate and the temperature.

2. Field equations

As mentioned above, for a description of the kinematical behaviour of nematic liquid crystals, it is necessary to specify the placement of each single material point **P** and the orientation of the molecule.

The orientation of a molecule is specified by means of a point lying on a spherical surface, on which the antipodal points are considered as representing the same molecule owing to the mirror symmetry of nematic molecules. This algebraic manifold is isomorphic to the projective plane Π^2. For each material point **P** of a body B, the *complete placement* is given thus by a mapping k defined by

$$k : B \to \mathscr{E}^3 \times \Pi^2 \tag{2.1}$$

such that

- the restriction of k at $\mathscr{E}^3$ gives the position of **P** in $\mathscr{E}^3$;
- the restriction of k at Π^2 characterises the orientation of the molecule centred at **P**.

In mechanics of fluids it is not important to fix a placement as a reference configuration but it is usual to refer the mechanical properties of the continuum to the current configuration. In the following, we consider the current configuration near to the reference configuration, thus it is possible to apply the linear theory of continuum mechanics. With this hypothesis, fixed a co-ordinate system, the apparent placement of each point of the continuum is indicated by a vector **x**: the ith component of which is x^i; thus, apparent motion is a time-parameterised family of placement

$$x^i(t). \tag{2.2}$$

The time derivative of position expresses the velocity of point **P**

$$\dot{x}^i = v^i = d_t x^i. \tag{2.3}$$

The acceleration is given by

$$a^i = d_t v^i = d^2_{tt} x^i. \tag{2.4}$$

The order parameter field identifying the direction of the molecule is defined by a vectorial function $\mathbf{d}(\mathbf{x}, t)$ of components

$$d^i(\mathbf{x}, t). \tag{2.5}$$

The ijth component of the gradient of **d** is indicated by

$$D^i_j = \partial_j d^i. \tag{2.6}$$

The microvelocity field is expressed by

$$\dot{d}^i = \partial_t d^i \tag{2.7}$$

and the microacceleration is given by

$$\ddot{d}^i = \partial^2_{tt} d^i. \tag{2.8}$$

The balance of momentum for such of kind of continuum is expressed by (see Capriz, 1989, 1995; Capriz and Biscari, 1994; Ericksen, 1962, 1991)

$$\begin{cases} \partial_j T^j_i + b_i = 0, & \text{in } B \\ \partial_j \mathscr{S}^j_i - z_i + \beta_i = 0, & \end{cases} \tag{2.9}$$

on the bulk, while on the boundary $\partial B(t)$

$$T^j_i m_j = t_i, \qquad \mathscr{S}^j_i m_j = \tau_i \tag{2.10}$$

m_i, being the components of the normal at the boundary and b_i, the body force density, β_i, the density of body forces acting on the microstructure (e.g. electromagnetic field), t_i, the macroscopic traction, τ_i, the generalised "traction" associated to the boundary data of the microstructure, T_i^j, the Cauchy stress, z_i, the self-interactions between elements of the microstructures, and $\mathscr{S}_i^j$, the microstress.

We may decompose the external body forces into their inertial (in) and non-inertial (ni) parts, $\boldsymbol{b} = \boldsymbol{b}^{\rm ni} + \boldsymbol{b}^{\rm in}$ and $\beta = \beta^{\rm ni} + \beta^{\rm in}$, by using a generalised form of D'Alambert's argument and we can identify $\mathbf{b}^{\rm in}$ with $-\rho\ddot{x}^i$, ρ, being the mass density and $\beta^{\rm in}$ with $-\left(\partial_{\dot{d}^i} k\left(d^j, \dot{d}^j\right)\right)^{\bullet} - \left(\partial_{d^i} k\left(d^j, \dot{d}^j\right)\right)$. $k(\mathbf{d}, \dot{\mathbf{d}})$ is the kinetic energy that can be attributed to the microstructure and is such that $k(\mathbf{d}, 0) = 0$ and $k\left(\cdot, \dot{\mathbf{d}}\right)$ is homogeneous in $\dot{\mathbf{d}}$ (Capriz and Virga, 1994). As a consequence, the balance of momentum becomes

$$\begin{cases} \partial_j T_i^j + b_i^{(\rm ni)} = \rho\delta_{ij}\ddot{x}^j, \\ \partial_j \mathscr{S}_i^j - z_i + \beta_i^{(\rm ni)} = (\partial_{\dot{d}^i} k(d^j, \dot{d}^j))^{\bullet} - (\partial_{d^i} k(d^j, \dot{d}^j)), \end{cases} \tag{2.11}$$

where δ_{ij} is the Kroeneker delta. The balance of momentum of momentum provides

$$e^i_{jk} T_i^k = -e^i_{jk} d^k z_i + (\partial_l(-e^i_{jk} d^k))\mathscr{S}_i^l, \tag{2.12}$$

where $\mathbf{e}$ is Ricci's tensor.

To complete the set of the field equations, the balance of mass, the balance of energy and Clausius–Duhem inequality must be introduced.

The balance of mass is given by

$$\dot{\rho} + \rho\partial_i \dot{x}^i = 0 \tag{2.13}$$

and the balance of energy is expressed by

$$\dot{\varepsilon} = T_i^j\left(\partial_j v^i\right) + \mathscr{S}_i^j \dot{D}_j^i + z_i \dot{d}^i + \partial_i q^i, \tag{2.14}$$

where ε is the density of the *internal energy* and $\mathbf{q}$ the heat flux of the ith component q^i. The Clausius–Duhem inequality can be written as

$$\dot{\Phi} - \mathbf{T}\cdot(\nabla\mathbf{v}) - \mathbf{S}\cdot\nabla\dot{\mathbf{d}} - \mathbf{z}\cdot\dot{\mathbf{d}} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta \leqslant 0, \tag{2.15}$$

where $\Phi = \varepsilon - \eta\theta$, with η the *entropy density* and θ the *temperature.* Neither heat sources nor entropy sources are considered.

For a perfect fluid with microstructure, we may use the mass density as a measure of compressibility of the liquid. Thus, Φ is considered as a function of the following variables:

$$\rho, \vartheta, \dot{\vartheta}, \partial_j\vartheta, d^i, D^i_j. \tag{2.16}$$

The additional requirements that the potential should be unaltered by superposed rigid body motion of the whole body and the symmetry properties of nematic liquid crystals allows us the substitution of $\partial_i\vartheta$ with the new variable $\lambda = 1/2(\partial_i\vartheta)(\partial_i\vartheta)$.

For a low range of temperature it is possible to take the potential Φ as a summation of two terms:

$$\Phi(\rho, \vartheta, \dot{\vartheta}, \lambda, d^i, D^i_j) = Q(\rho, \vartheta, \dot{\vartheta}, \lambda) + E(d^i, D^i_j), \tag{2.17}$$

where Q is a positive definite second order polynomial that takes into account only the energetic properties of the liquid and the term E is the Oseen–Frank potential

$$E = \tfrac{1}{2}K_{11}\left(D^i_i\right)^2 + \tfrac{1}{2}K_{22}(e^k_{ij} d^i D^j_k)^2 + \tfrac{1}{2}K_{33}(\delta_{il} d^j d^k D^i_k D^l_j) - (K_{22} + K_{24})(\delta^k_i\delta^l_j - \delta^l_i\delta^k_j)D^i_k D^j_l. \tag{2.18}$$

The field linked to K_{11} is said to be a *splay field* and K_{11} is the *splay modulus* likely, K_{22} is said to be the *bend modulus*, K_{33} the *twist modulus* and $(K_{22}+K_{24})$ the *saddle-splay modulus* (see Virga, 1994). For a low range of temperature the Frank constant are independent of temperature.

The application of standard arguments involving the validity of Clausius–Duhem inequality for every choice of the velocity field (see Capriz, 1989), according to constitutive choice (2.17) and (2.18), leads to

$$T_i^j = -\rho^2 \partial_\rho Q \delta_i^j - \rho \partial_l Q \delta^{il} \delta_i^j (\partial_j \vartheta)(\partial_l \vartheta) + \rho D_k^j \partial_{D_k^i} E, \tag{2.19}$$

$$\mathscr{S}_i^j = \partial_{D_j^i} E, \tag{2.20}$$

$$z_i = \partial_{d^i} E, \tag{2.21}$$

$$\eta = -\partial_\theta Q. \tag{2.22}$$

The heat flux vector $\mathbf{q}$ is related to the gradient of temperature by

$$q^i = K^{il}(\rho, \vartheta, \dot{\vartheta}, \lambda, d^i, D_j^i)\, \partial_l \vartheta \tag{2.23}$$

with $\boldsymbol{K}$ an appropriate positive definite second order tensor valued function. We assume in the following the possibility to decompose $\boldsymbol{K}$ in two terms, namely

$$K^{il}(\rho, \vartheta, \dot{\vartheta}, \lambda, d^i, D_j^i) = \bar{K}^{il}(\rho, \vartheta, \dot{\vartheta}, \lambda) + \hat{K}^{il}(d^i, D_j^i), \tag{2.24}$$

where $\bar{\boldsymbol{K}}$ depends only on the properties of liquid and $\hat{\boldsymbol{K}}$ the characteristics of crystals.

3. Discontinuity surfaces

Let us consider a moving surface S defined in Euclidean space by a function Σ such that

$$\Sigma(x^i, t) = 0, \quad i = 1, 2, 3. \tag{3.1}$$

Assume also that the surface intersects the body $\mathscr{B}$ during a time interval $[t_0, t_1]$ and we assume that Σ is continuous with its first derivatives. Let us define for every point $\mathbf{x}$ belonging to S the unit normal $n_i = \partial_i \Sigma / \sqrt{\delta^{ij} \partial_i \Sigma \partial_j \Sigma}$ and the normal velocity $u = \dot{\mathbf{x}} \cdot \mathbf{n}$ given by

$$u = -\frac{\partial_t \Sigma}{\sqrt{\delta^{ij} \partial_i \Sigma \partial_j \Sigma}}, \tag{3.2}$$

where ∂_t means the time partial derivative. Let us assume that the surface S divides the body in two disjoint sub-bodies $\mathscr{B}^+$ with the outward normal, and $\mathscr{B}^-$. For every field ℓ on $\mathscr{B} \times \mathrm{R}$ for every $\mathbf{x} \in S$, it is possible to define the limits

$$\ell^\pm = \lim_{\delta \to 0} \ell(\mathbf{x} \pm \delta \mathbf{n}, t), \qquad \ell^\pm = \lim_{\delta \to 0} \ell(\mathbf{x}, t \mp \delta). \tag{3.3}$$

In the following we will use only the former definition.

The *jump*, across S, of the field ℓ is the difference:

$$[\ell] = \ell^+ - \ell^-. \tag{3.4}$$

Given two fields ℓ and $\mathscr{g}$, $[\ell \mathscr{g}] = -[\ell][\mathscr{g}] + \mathscr{g}^+[\ell] + \ell^+[\mathscr{g}]$ (for a detailed treatment of discontinuity surfaces see Manacorda (1979); Šilhavý (1997)).

Acceleration waves are discontinuity surfaces characterised by

$$[x^i] = 0, \quad [d^i] = 0, \quad \lfloor \dot{\vartheta} \rfloor = 0, \quad [\rho] = 0, \quad \lfloor \dot{x}^i \rfloor = 0, \quad [\dot{d}^i] = 0, \quad [D_i^j] = \mathbf{0}, \tag{3.5}$$

while $\lfloor d_{tt}^2 x^i \rfloor \neq 0$ and $\lfloor \partial_{tt}^2 d^i \rfloor \neq 0$.

If the heat flux vector is continuous across the surface S, the wave is *homothermal*; for this kind of wave, the gradient of temperature and its first time derivative are continuous across S, namely

$$\lfloor \dot{\vartheta} \rfloor = 0, \qquad [\partial_i \vartheta] = \mathbf{0}. \tag{3.6}$$

In the treatment of discontinuity waves, Hadamard theorem plays a fundamental role; it links in fact the jump of the gradient of a function with the gradient of the jump and time derivative of the same function. More precisely,

$$[\partial_i \ell] = \partial_i [\ell] - \frac{n_i}{u} [\partial_t \ell]. \tag{3.7}$$

In what follows *we assume that the kinetic energy pertaining the order parameter field is quadratic*, i.e. there exists a second order tensor $\mathscr{J}$ such that

$$k(\mathbf{d}, \dot{\mathbf{d}}) = \tfrac{1}{2} \dot{\mathbf{d}} \mathscr{J} \dot{\mathbf{d}}. \tag{3.8}$$

$\mathscr{J}$ accounts for the characteristics of inertia of the slender molecules.

The balance equations, reduced at the discontinuity surfaces, assume the following form:

conservation of mass:

$$[\dot{\rho}] = -\rho \lfloor v_i^i \rfloor, \tag{3.9}$$

conservation of momentum:

$$\left[\partial_j T_i^j\right] = \rho \delta_{ij} \left[\dot{v}^j\right], \tag{3.10}$$

conservation of micromomentum:

$$\left[\partial_j \mathscr{S}_i^j\right] = \mathscr{J}_{ij} [\ddot{d}^j], \tag{3.11}$$

conservation of energy:

$$[\partial_i q^i] = [\dot{\varepsilon}]. \tag{3.12}$$

4. Propagation of homothermal acceleration waves

The problem of propagation and growth of an acceleration wave in a thermoelastic continuum with microstructure and in particular in a liquid crystal is not so simple because in its formulation a great number of variables come in. It is possible to divide such an analysis into three steps. At the first step it is possible to determine the velocities and the directions of propagation of the surface within the medium, at the second one it is necessary to analyse the variation of the wave configuration during the propagation with particular attention to curvature tensor of the surface and it is possible to write a differential equation ruling the growth and the decay of the amplitude of acceleration jumps across the surface, during the motion.

By applying Hadamard theorem to Eqs. (3.9)–(3.12) one obtains

$$\begin{cases} [\dot\rho] = -\rho[\partial_i \dot x^i], \\ \left[\partial_j T_i^j\right] = \rho\delta_{ij}[\dot v^j], \\ \left[\partial_j \mathscr{S}_i^j\right] = \mathscr{I}_{ij}[\ddot d^j], \\ \rho\vartheta[\dot\eta] = [\partial_i q^i]. \end{cases} \tag{4.1a–d}$$

The application of Hadamard theorem to Eq. (4.1) gives

$$[\rho] = \rho\frac{n_i}{u}[\ddot x^i], \tag{4.2}$$

$$[\dot T_i^j]n_j = -\rho\delta_{ij}u[\ddot x^j], \tag{4.3}$$

$$[\dot{\mathscr{S}}_i^j]n_j = -\mathscr{I}_{ij}[\ddot d^j]. \tag{4.4}$$

From the fourth equation, with the application of homothermal conditions, it follows that

$$u\rho\vartheta[\dot\eta] = -[\dot q^i]n_i. \tag{4.5}$$

Now, by developing the time derivative of the stress tensor, namely

$$\dot T_j^i = \partial_\rho T_j^i\dot\rho + \partial_{d^k}T_j^i\dot d^k + \partial_{D_l^j}T_j^i\dot D_l^j + \partial_\vartheta T_j^i(\partial_t\vartheta) + \partial_{\dot\vartheta}T_j^i\left(\partial_{tt}^2\vartheta\right) + \partial_\lambda T_j^i(\partial_t\lambda) \tag{4.6}$$

and evaluating the jump across the discontinuity surface, we obtain

$$\lfloor \dot T_j^i\rfloor = \partial_\rho T_j^i[\dot\rho] + \partial_{D_l^k}T_j^i\lfloor \dot D_l^k\rfloor + \partial_{\dot\vartheta}T_j^i\lfloor\partial_{tt}^2\vartheta\rfloor. \tag{4.7}$$

By using Hadamard theorem, in Eq. (4.7) the term $\left[\dot D_i^j\right]$ may be substituted by $-(n_i/u)\left[\partial_{tt}^2 d^j\right]$.

Applying the same procedure to Eq. (4.1c,d) and making use of relations (2.20) and (2.21), we get the following system:

$$\mathscr{A}(u)\,[\dot{\boldsymbol\xi}] - \rho u\mathscr{I}[\dot{\boldsymbol\xi}] = \mathbf{0} \tag{4.8}$$

collecting in a single vector the unknowns with the following position: $\lfloor\dot\xi^1\rfloor = [\dot\rho]$, for $i = 2,3,4$ $\lfloor\dot\xi^i\rfloor = \lfloor\ddot x^i\rfloor$, for $i = 5,6,7$; $\lfloor\dot\xi^i\rfloor = \lfloor\partial_{tt}^2 d^i\rfloor$ and $\lfloor\dot\xi^8\rfloor = \lfloor\partial_{tt}^2\vartheta\rfloor$, with:

$$\mathscr{A}_{1i} = -\rho\frac{n_i}{u}, \quad i = 2,3,4, \tag{4.9}$$

$$\begin{aligned}\mathscr{A}_{i1} = -\{&(2\rho\partial_\rho Q + \rho^2\partial_{\rho\rho}^2 Q)\delta_i^p + (\rho\partial_{\lambda\rho}^2 Q + \partial_\lambda Q)\,\delta^{pr}\delta_i^q\partial_r\vartheta\partial_q\vartheta + \delta_i^p D_q^j D_r^k(K_{11}\delta_j^q\delta_k^r + K_{22}e_{mj}^q e_{lk}^r d^m d^l \\ &+ K_{33}\delta_{ik}\delta_l^q\delta_m^r d^m d^l - (K_{22}+K_{24})(\delta_j^q\delta_k^r - \delta_j^r\delta_k^q))\}\frac{n_p}{u}, \quad i = 2,3,4,\end{aligned} \tag{4.10}$$

$$\begin{aligned}\mathscr{A}_{ij} = -\rho\frac{n_p n_q}{u}\,&\delta_i^p D_q^r(K_{11}\delta_j^q\delta_k^r + K_{22}e_{mj}^q e_{lk}^r d^m d^l + K_{33}\delta_{ik}\delta_l^q\delta_m^r d^m d^l \\ &- (K_{22}+K_{24})\,(\delta_j^q\delta_k^r - \delta_j^r\delta_k^q)), \quad i,j = 2,3,4,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\mathscr{A}_{ij} = -\rho\frac{n_i n_r}{u}D_q^k(K_{11}\delta_k^q\delta_j^r + K_{22}e_{mk}^q e_{lj}^r d^m d^l + K_{33}\delta_{jk}\delta_l^q\delta_m^r d^m d^l - (K_{22}+K_{24})\,(\delta_k^q\delta_j^r - \delta_k^r\delta_j^q)), \\ i = 2,3,4 \quad j = 5,6,7,\end{aligned} \tag{4.12}$$

$$\mathscr{A}_{i4} = n_p\Big(\Big(- \partial^2_{\rho\dot\vartheta}Q + \frac{n_r}{u}(\partial^2_{\rho\lambda}Q\partial_r\vartheta + \partial^2_{\lambda\lambda}Q\partial_r\vartheta)\Big)\delta^p_i - \rho\partial^2_{\lambda\dot\vartheta}Q\delta^{pr}\delta^q_i\partial_r\vartheta\partial_q\vartheta$$
$$+ \frac{n_m}{u}\left(\partial^2_{\lambda\lambda}Q\partial_r\vartheta\partial_m\vartheta + \rho\partial_r Q\right)\delta^{pr}\delta^q_i\partial_q\vartheta\Big), \quad i = 2,3,4, \tag{4.13}$$

$$\mathscr{A}_{i1} = \left(K_{11}\delta^q_i\delta^r_k + K_{22}e^q_{mi}e^r_{lk}d^m d^l + K_{33}\delta_{ik}\delta^q_l\delta^r_m d^m d^l - (K_{22}+K_{24})\left(\delta^q_i\delta^r_k - \delta^r_i\delta^q_k\right)\right)D^j_q\frac{n_r}{u}, \quad i = 5,6,7, \tag{4.14}$$

$$\mathscr{A}_{ij} = -\rho\Big(K_{11}\delta^q_i\delta^r_j + K_{22}e^q_{mi}e^r_{lj}d^m d^l + K_{33}\delta_{ij}\delta^q_l\delta^r_m d^m d^l - (K_{22}+K_{24})\Big(\delta^q_i\delta^r_j - \delta^r_i\delta^q_j\Big)\Big)\frac{n_r n_q}{u},$$
$$i,j = 5,6,7, \tag{4.15}$$

$$\mathscr{A}_{81} = \rho u\vartheta\partial_\rho Q - \frac{n_i}{u}\partial_\rho K^{ir}\partial_r\vartheta, \tag{4.16}$$

$$\mathscr{A}_{8i} = \frac{n_j n_r}{u^2}\partial_{D^i_r}K^{jp}\partial_p\vartheta, \quad i = 5,6,7, \tag{4.17}$$

$$\mathscr{A}_{88} = -\rho\partial^2_{\dot\vartheta\dot\vartheta}\Psi + \frac{n_i}{u}\partial_{\dot\theta}K^{ip}\partial_p\vartheta - \frac{n_i n_q}{u^2}K^{iq}, \tag{4.18}$$

where $\mathscr{I} = \mathrm{diag}(0, \mathbf{I}, \mathscr{J}, 0)$, $\mathbf{I}$ is the identity matrix in $\mathscr{E}^3$.

Proposition 4.1. *The velocities of propagation of an homothermal acceleration wave are the values of u that allow the appearance of non-trivial solutions of Eq.* (4.9), *to each non-trivial solution a direction of propagation in the space of the unknowns corresponds, the projection of such direction in the sub-spaces of jumps of macroaccelerations and microaccelerations provides the directions of propagation of the discontinuities in the Euclidean space.*

Proof. From Rouché theorem, the homogeneous system (4.9), admits non-trivial solutions if and only if

$$\det(\mathscr{A}(u) - \rho u\mathscr{I}) = 0. \tag{4.19}$$

Moreover we may write

$$[\dot{\xi}] = \sum_L \boldsymbol{\sigma}_L \mathbf{r}_L, \tag{4.20}$$

where $\boldsymbol{\sigma}_L$ is the amplitude of the jump and $\mathbf{r}_L$ is a unit eigenvector. L marks the Lth velocity of propagation. Taking out the singles parts of vector, we obtain the direction of propagation of mechanical and thermal accelerations in the Euclidean space, and similarly, the jumps of the mass velocity.

Eq. (4.19) is the *characteristic equation* of the system (4.8). □

A general result, which is useful in deriving the following developments, is the *lemma of bicharacteristic directions*.

Lemma 4.1. (Courant and Hilbert, 1962). *At a fixed point* $\boldsymbol{x}$, *consider the characteristic matrix* $\mathscr{K}$ *of a hyperbolic system* $n \times n$ *of partial differential equations and take* $\mathscr{K}$ *as a function of* $\boldsymbol{n} = \nabla\Sigma/|\nabla\Sigma|$. *Assume also that rank* $(\mathscr{K}) = n - 1$. *Then, the relation that gives the differentiation along the rays of bicharacteristics is*

$$\boldsymbol{x}' = \boldsymbol{l}^t\frac{\partial\mathscr{K}}{\partial\boldsymbol{n}}\boldsymbol{r}, \tag{4.21}$$

where the prime denotes differentiation along the ray with respect to the parameter on the curve, and **l** *and* **r** *are the left and the right vectors belonging to the kernel of the matrix* $\mathscr{K}$.

5. Motion of a singular surface

Usually, during the propagation, the wave front is subjected to changes in shape. The study of evolution of the shape of wave front plays a fundamental role where connected with the evolution of jumps.

Each surface solution of Eq. (4.19) may be expressed in parametric form by its Gaussian co-ordinates and the parameter $z^3 = t - t_0$ as

$$\mathbf{x} = \hat{\mathbf{x}}(z^1, z^2, z^3), \qquad \mathbf{x} \in S(t) \cap \mathbf{B}. \tag{5.1}$$

The gradient of previous transformation is given by

$$\frac{\partial \mathbf{x}}{\partial z^p} = \mathbf{a}_p \quad p = 1, 2, \tag{5.2}$$

where $\mathbf{a}_p$ are vectors of covariant basis of the singular surface and

$$\frac{\partial \mathbf{x}}{\partial z^3} = \mathbf{b} \tag{5.3}$$

is the *ray velocity*.

The Jacobian determinant of the transformation (5.1) is defined by

$$J = \det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{z}}\right) = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{b} = a^{1/2} v \tag{5.4}$$

being [1]

$$v = u_N(\mathbf{x}, \mathbf{n}, \rho, \mathbf{d}, \mathbf{D}, \vartheta, \dot{\vartheta}, \lambda, t), \tag{5.5}$$

the velocity of propagation of the discontinuity surface. From Eq. (5.5), the following proposition follows:

Proposition 5.1. *If propagation of the discontinuity surface occurs in a homogeneous material, deformed homogeneously with uniform temperature and order parameters, the velocity of propagation of a wave front is constant and the surface propagates along straight trajectories.*

By derivation of Eq. (5.5), the variation of velocity of propagation along a ray is expressed by

$$\begin{aligned} v' &= \frac{\mathrm{d}v}{\mathrm{d}z^3} = \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \frac{\partial u_N}{\partial t} + \frac{\partial u_N}{\partial \mathbf{n}}\frac{\partial \mathbf{n}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial u_N}{\partial \rho}\frac{\partial \rho}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial u_N}{\partial \mathbf{d}}\frac{\partial \mathbf{d}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial u_N}{\partial \mathbf{D}}\frac{\partial \mathbf{D}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial u_N}{\partial \vartheta}\frac{\partial \vartheta}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial u_N}{\partial \dot{\vartheta}}\frac{\partial \dot{\vartheta}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} \\ &\quad + \frac{\partial u_N}{\partial \lambda}\frac{\partial \lambda}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial u_N}{\partial t} + v\mathbf{n} \cdot \nabla u_N. \end{aligned} \tag{5.6}$$

[1] In what follows, low capital indexes are enumerative only, they must not be summed.

The use of Euler's formula for the derivative of a determinant, namely $J' = J\mathrm{tr}[(\partial \mathbf{z}/\partial \mathbf{x})(\partial \mathbf{x}'/\partial \mathbf{z})]$, implies the following proposition:

Proposition 5.2. *The Jacobian determinant of the transformation* (5.1) *varies along a ray in accordance with the following expression*:

$$\begin{aligned}\frac{J'}{J} &= \frac{\mathrm{d}(\lg J)}{\mathrm{d}u^3}\\ &= \mathrm{tr}\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \mathbf{x}} + \frac{1}{v}\frac{\partial u_N}{\partial t} - \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}^2}\mathsf{B}\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \rho}\frac{\partial \rho}{\partial z^p}\otimes z^p\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \mathbf{d}}\frac{\partial \mathbf{d}}{\partial z^p}\otimes z^p\right)\\ &\quad + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \mathbf{D}}\frac{\partial \mathbf{D}}{\partial z^p}\otimes z^p\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \vartheta}\frac{\partial \vartheta}{\partial z^p}\otimes z^p\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \dot{\vartheta}}\frac{\partial \dot{\vartheta}}{\partial z^p}\otimes z^p\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \lambda}\frac{\partial \lambda}{\partial z^p}\otimes z^p\right),\end{aligned} \tag{5.7}$$

where B *is the surface curvature tensor of* S

$$\mathsf{B} = \mathsf{B}^p_q \mathbf{a}_p \otimes \mathbf{a}^q = -\frac{\partial \mathbf{n}}{\partial z^p}\otimes \mathbf{a}^p. \tag{5.8}$$

To obtain Eq. (5.7) we derive Eq. (5.5) with respect to ray parameter u^3 and, using expression (5.6).

Therefore, when the analytic expressions of components of curvature tensor B are known, as functions of the ray parameter, Eq. (5.7) may be solved by quadrature.

A direct derivation of the vectors of covariant basis gives us

$$\mathbf{a}'_p = -\frac{\partial^2 u_N}{\partial \mathbf{n}^2}\mathsf{B}^q_p \mathbf{a}_q + \left(\frac{\partial^2 u_N}{\partial \mathbf{x}\,\partial \mathbf{d}}\frac{\partial \mathbf{d}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{x}\,\partial \mathbf{D}}\frac{\partial \mathbf{D}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \rho}\frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \vartheta}\frac{\partial \vartheta}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \dot{\vartheta}}\frac{\partial \dot{\vartheta}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \lambda}\frac{\partial \lambda}{\partial \mathbf{x}}\right)\mathbf{a}_p. \tag{5.9}$$

Proposition 5.3. *The area element varies, during the motion, according to the following expression*:

$$\begin{aligned}\left(\lg a^{1/2}\right)' &= \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}^2}\mathsf{B}\right) + \mathrm{tr}\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \mathbf{x}} - \mathbf{n}\cdot\frac{\partial u_N}{\partial \mathbf{x}} + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \theta}\frac{\partial \vartheta}{\partial z^p}\otimes z^p\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \rho}\frac{\partial \rho}{\partial z^p}\otimes z^p\right)\\ &\quad + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \mathbf{d}}\frac{\partial \mathbf{d}}{\partial z^p}\otimes z^p\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \vartheta}\frac{\partial \vartheta}{\partial z^p}\otimes z^p\right) + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \dot{\vartheta}}\frac{\partial \dot{\vartheta}}{\partial z^p}\otimes z^p\right)\\ &\quad + \mathrm{tr}\left(\frac{\partial^2 u_N}{\partial \mathbf{n}\,\partial \lambda}\frac{\partial \lambda}{\partial z^p}\otimes z^p\right).\end{aligned} \tag{5.10}$$

Eq. (5.10) may be obtained from a differentiation of definition $J = a^{1/2}v$ and the use of (5.6).

As above mentioned, the characterisation of the wave front shape may be expressed with the component of surface curvature tensor B. By direct calculation, we obtain the following proposition:

Proposition 5.4. *The variation along a ray of each component of surface curvature tensor is subjected to the following restrictions*:

$$
\begin{aligned}
\mathsf{B}_p^{\prime q} = {} & \mathsf{B}_p^r \mathsf{B}_r^s \operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{n}^2} \right) (\mathbf{a}_s \otimes \mathbf{a}^q) \right\} + \mathsf{B}_p^q \mathbf{n} \cdot \frac{\partial u_N}{\partial \mathbf{x}} - \mathsf{B}_p^r \operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \mathbf{x}} \right) (\mathbf{a}_s \otimes \mathbf{a}^q + \mathbf{a}^q \otimes \mathbf{a}_s) \right\} \\
& + \left(\frac{\partial^2 u_N}{\partial \mathbf{x}^2} \right) \operatorname{tr}\left(\mathbf{a}_p \otimes \mathbf{a}^q \right) - \mathsf{B}_p^r \operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \mathbf{d}} \frac{\partial \mathbf{d}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \mathbf{D}} \frac{\partial \mathbf{D}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \rho} \frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \vartheta} \frac{\partial \vartheta}{\partial \mathbf{x}} \right. \right. \\
& \left. \left. + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \dot{\vartheta}} \frac{\partial \dot{\vartheta}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \lambda} \frac{\partial \lambda}{\partial \mathbf{x}} \right) (\mathbf{a}_r \otimes \mathbf{a}^q) \right\} + \left(\frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{d}} \frac{\partial \mathbf{d}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{D}} \frac{\partial \mathbf{D}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \rho} \frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \vartheta} \frac{\partial \vartheta}{\partial \mathbf{x}} \right. \\
& \left. + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \dot{\vartheta}} \frac{\partial \dot{\vartheta}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \lambda} \frac{\partial \lambda}{\partial \mathbf{x}} \right) \operatorname{tr}\left(\mathbf{a}_p \otimes \mathbf{a}^q \right). \qquad (5.11)
\end{aligned}
$$

To characterise locally the wave front, it is sufficient to define the invariants of the surface curvature tensor, i.e. the mean curvature $\mathsf{b} = 1/2 \operatorname{tr} \mathsf{B}$ and the Gaussian curvature $B = \det \mathsf{B}$. They change in accordance with

$$
\begin{aligned}
2\mathsf{b} = {} & \mathsf{B}_p^{\prime p} \\
= {} & \operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{n}^2} \right) \mathsf{B}^2 \right\} + -(\operatorname{tr} \mathsf{B}) \mathbf{n} \cdot \frac{\partial u_N}{\partial \mathbf{x}} - 2\operatorname{tr}\left\{ \left(\frac{\partial^2 U_N}{\partial \mathbf{n} \, \partial \mathbf{x}} \right) \mathsf{B} \right\} + \operatorname{tr}\left\{ \left(\frac{\partial^2 U_N}{\partial \mathbf{x}^2} \right) \left(\mathbf{a}_p \otimes \mathbf{a}^q \right) \right\} \\
& + \operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{d}} \frac{\partial \mathbf{d}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{D}} \frac{\partial \mathbf{D}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \rho} \frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \vartheta} \frac{\partial \vartheta}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \dot{\vartheta}} \frac{\partial \dot{\vartheta}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \lambda} \frac{\partial \lambda}{\partial \mathbf{x}} \right) \mathsf{B} \right\} \\
& + \operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{d}} \frac{\partial \mathbf{d}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{D}} \frac{\partial \mathbf{D}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \rho} \frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \vartheta} \frac{\partial \vartheta}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \dot{\vartheta}} \frac{\partial \dot{\vartheta}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \lambda} \frac{\partial \lambda}{\partial \mathbf{x}} \right) \left(\mathbf{a}_p \otimes \mathbf{a}^q \right) \right\}, \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
(\lg(|\mathsf{B}|))' = {} & \frac{|\mathsf{B}|'}{|\mathsf{B}|} \\
= {} & \operatorname{tr}\left\{ \left(\frac{\partial^2 U_N}{\partial \mathbf{n}^2} \right) \mathsf{B} \right\} + 4\mathbf{n} \cdot \frac{\partial U_N}{\partial \mathbf{x}} - 2\operatorname{tr}\left\{ \frac{\partial^2 U_N}{\partial \mathbf{n} \, \partial \mathbf{x}} \right\} + \operatorname{tr}\left\{ \left(\frac{\partial^2 U_N}{\partial \mathbf{x}^2} \right) \mathsf{B}^{-1} \right\} \\
& + -\operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{d}} \frac{\partial \mathbf{d}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{D}} \frac{\partial \mathbf{D}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \rho} \frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \vartheta} \frac{\partial \vartheta}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \dot{\vartheta}} \frac{\partial \dot{\vartheta}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \lambda} \frac{\partial \lambda}{\partial \mathbf{x}} \right) \right\} \\
& + \operatorname{tr}\left\{ \left(\frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{d}} \frac{\partial \mathbf{d}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{x} \, \partial \mathbf{D}} \frac{\partial \mathbf{D}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \rho} \frac{\partial \rho}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \vartheta} \frac{\partial \vartheta}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \dot{\vartheta}} \frac{\partial \dot{\vartheta}}{\partial \mathbf{x}} + \frac{\partial^2 u_N}{\partial \mathbf{n} \, \partial \lambda} \frac{\partial \lambda}{\partial \mathbf{x}} \right) (\mathsf{B}^{-1}) \right\}. \\
& \qquad (5.13)
\end{aligned}
$$

6. The evolution equation of the amplitude

If the values of the jumps of macroacceleration and microacceleration, second time derivative of the temperature and the time derivative of mass density, are assigned at the initial wave front, it is possible to deduce an equation regulating the variation of the acceleration jumps during the propagation.

To obtain such an evolution equation, we follow the usual procedure for the solution of hyperbolic system, presented for linear systems in Courant and Hilbert (1962) and extended by Varley and Cumberbatch (1965) to quasilinear systems.

However, the starting point for the determination of the evolution equation is the calculation of the time derivatives of balance equations across the surface S, namely

$$\begin{cases} [\ddot{\rho}] = -[(\rho\partial_i\dot{x}^i)^\bullet] \\ [\partial_j\dot{T}_i^j] = \delta_{ij}[(\rho\ddot{x}^j)^\bullet] \\ [\partial_j\dot{\mathscr{S}}_i^j] - [\dot{z}_i] = \mathscr{I}_{ij}[\dddot{d}^j] \\ [\ddot{\varepsilon}] = [(\partial_i q^i)^\bullet] \end{cases} \tag{6.1}$$

and of the time derivative of the Fourier law (2.23)

$$[\dot{q}^i] = [(K^{ij}(\rho,\vartheta,\dot{\vartheta},\lambda,d^k,D_l^k)\partial_j\vartheta)^\bullet]. \tag{6.2}$$

Proposition 6.1. *The transport equation of jump is expressed by the Bernoulli equation*

$$\sigma' = b(z^3)\sigma^2 - c(z^3)\sigma, \tag{6.3}$$

where σ is the scalar amplitude of the jumps of $\dot{\xi}$ and the coefficients $b(z^3)$ e $c(z^3)$ pick up the whole thermodynamic state ahead of the wave, the rheological properties of material and the instantaneous shape of the wave front.

Proof. By evaluating the explicit expression of the time derivatives in Eq. (6.1) and applying Hadamard theorem at each jump of spatial derivatives, we obtain a system that may by written in compact form as follows:

$$(\mathscr{A} - \rho u\mathscr{I})_{ij}\left[\ddot{\xi}^j\right] + \mathscr{C}_{ij}\left[\dot{\xi}^j\right] + \mathscr{B}_{ijk}\left[\dot{\xi}^j\right]\left[\dot{\xi}^k\right] + \tilde{\partial}\left(\partial_{\mathbf{n}}\mathscr{A}_{ij}[\dot{\xi}^j]\right) = 0, \tag{6.4}$$

where the matrices $\mathscr{A}$ and $\mathscr{I}$ are given in Section 4, and the tensorial coefficients $\mathscr{B}$ and $\mathscr{C}$ are in Appendix A. The symbol $\tilde{\partial}$ means partial derivative with respect to the ith co-ordinate of the terms $\mathscr{A}$, $\mathbf{n}/u$, $[\dot{\boldsymbol{\xi}}]$. Recalling the position $[\dot{\boldsymbol{\xi}}] = \sigma_L\mathbf{r}_L$, we may assume $[\ddot{\boldsymbol{\xi}}] = \sum_M \gamma_M\mathbf{r}_M$ where γ_M marks the jumps of time derivatives of $\dot{\boldsymbol{\xi}}$. Consequently

$$(\mathscr{A} - \rho u\mathscr{I})\sum_M \gamma_M\mathbf{r}_M + \sigma_L\mathscr{C}\mathbf{r}_L + \sigma_L^2\mathscr{B}\mathbf{r}_L\otimes\mathbf{r}_L + \tilde{\partial}(\partial_{\mathbf{n}}\mathscr{A}\sigma_L\mathbf{r}_L) = \mathbf{0}, \tag{6.5}$$

where the terms γ_L appear with σ_L. Nevertheless, with a left scalar multiplication for $\mathbf{l}_L$ we eliminate the indetermination due to presence of γ_L, because $\mathbf{l}_L(\mathscr{A} - \rho u\mathscr{I}) = \mathbf{0}$, and we obtain the scalar equation

$$\sigma_L\mathbf{l}_L\cdot\mathscr{C}\mathbf{r}_L + \sigma_L^2\mathbf{l}_L\cdot\mathscr{B}\mathbf{r}_L\otimes\mathbf{r}_L + \sigma_L^{-1}\partial_\alpha\left(\mathbf{l}_L\cdot\partial_{\mathbf{n}}\mathscr{A}\mathbf{r}_L\sigma_L^2\right) = 0. \tag{6.6}$$

Given a single mode of propagation, we may apply the lemma on bicharacteristic directions, expressed in Lemma 4.1, and may write, eliminating the unnecessary lower index L,

$$\mathscr{c}\sigma^2 + \mathscr{b}\sigma^3 + \partial_\alpha\left(\rho\dot{\xi}'\sigma^2\right) = 0, \tag{6.7}$$

where

$$\mathscr{c} = \mathbf{l}\cdot\mathscr{C}\mathbf{r}, \tag{6.8}$$

$$\mathscr{b} = \mathbf{l}\cdot\mathscr{B}(\mathbf{r}\otimes\mathbf{r}). \tag{6.9}$$

Using Euler's formula for the derivative of a determinant, we obtain $\sigma^{-1}\partial_\alpha(\rho\dot{\xi}'\sigma^2) = (J\sigma)^{-1}(\rho J\sigma^2)'$, where the symbol $(\ )'$ notes the derivation with respect to the ray parameter. Eq. (6.7) may be reduced to the normal form (6.3) with the following positions:

$$b(z^3) = -\frac{\mathscr{b}}{2\rho}, \tag{6.10}$$

$$c(z^3) = \left(\frac{\mathscr{c}}{2\rho} + \frac{1}{2} \frac{(\rho J)'}{\rho J} \right). \tag{6.11}$$

The coefficient b depends on the rheological properties of material in which the wave propagates (see Appendix A). In this particular case, it depends on the mass density, the Franck's constants and the thermal conductivity tensor, while coefficient c picks up the response of material ahead of the wave and the instantaneous configuration of the wave front.

If the initial amplitude σ_0 is assigned and the whole thermodynamic response of the material is known, it is possible to integrate Eq. (6.3). The solution is expressed by

$$\sigma(z^3) = \frac{\mathrm{e}^{-\int_0^{z^3} c(\varsigma)\,\mathrm{d}\varsigma}}{\frac{1}{\sigma_0} - \int_0^{z^3} b(\varsigma)\mathrm{e}^{-\int_0^{\varsigma} c(\bar{\varsigma})\,\mathrm{d}\bar{\varsigma}}}. \tag{6.12}$$

A full description of local and global behaviour of Eq. (6.3) may be found in Bailey and Chen (1971a,b).

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Appendix A

Coefficients of tensors of Eq. (6.3)

$$\mathscr{C}_{1j} = -\rho \frac{n_i}{u} \left(\partial_j v^i\right)^+ + \frac{n_j}{u} (\dot{\rho})^+, \quad j = 2, 3, 4,$$

$$\mathscr{B}_{1jk} = -\rho \frac{n_j n_k}{u^2}, \quad j, k = 2, 3, 4,$$

$$\begin{aligned}
\mathscr{C}_{i1} = {} & 2\Big(\partial_\rho Q \frac{n_i}{u} (\dot{\rho})^+ - \partial_\rho Q (\partial_i \rho)^+ + \rho \partial^2_{\rho\rho} Q \frac{n_i}{u} (\dot{\rho})^+ - \rho \partial^2_{\rho\rho} Q (\partial_i \rho)^+ + \rho \partial^2_{\rho\vartheta} Q (\partial_i \vartheta) - \rho \partial^2_{\rho\dot{\vartheta}} Q \left(\partial_i \dot{\vartheta}\right)^+ \\
& - \rho \partial^2_{\rho\lambda} Q (\partial_i \lambda) \Big) - 2\rho \partial^2_{\rho\rho} Q \Big(-\frac{n_i}{u} (\dot{\rho})^+ + (\partial_i \rho)^+ \Big) + 2\rho \partial^2_{\rho\vartheta} Q \frac{n_i}{u} (\dot{\vartheta}) + 2\rho \partial^2_{\rho\dot{\vartheta}} Q \frac{n_i}{u} (\ddot{\vartheta})^+ + 2\rho \partial^2_{\rho\lambda} Q \frac{n_i}{u} (\dot{\lambda}) \\
& - \partial_\lambda Q \frac{n_i}{u} + \dot{\rho} \partial^2_{\lambda\rho} Q \frac{n_i}{u} \lambda - \partial^2_{\rho\lambda} Q \Big(\lambda \Big(-\frac{n_i}{u} (\dot{\rho})^+ + (\partial_i \rho)^+ \Big) + \rho (\partial_i \lambda) \Big) + \partial^2_{\lambda\vartheta} Q \frac{n_i}{u} \lambda \dot{\vartheta} + \left(\partial^2_{\lambda\dot{\vartheta}} Q + \partial^2_{\lambda\lambda} Q \right) \frac{n_i}{u} \lambda (\ddot{\vartheta})^+ \\
& + \left(\partial_\lambda Q + \partial^2_{\lambda\rho} Q \right) \frac{n_i}{u} \dot{\lambda} + D^k_q \big(K_{11} \delta^p_i \delta^q_k + K_{22} e^p_{mi} e^q_{lk} d^m d^l + K_{33} \delta_{ik} \delta^p_l \delta^q_m d^m d^l \\
& - (K_{22} + K_{24}) \big(\delta^p_i \delta^q_k - \delta^q_i \delta^p_k \big) \big) \left(\partial_p D^j_p \right)^+ + D^j_p \big(K_{11} \delta^p_i \delta^q_k + K_{22} e^p_i e^q_k d\, d + K_{33} \delta_{ik} \delta^p \delta^q d\, d \\
& - (K_{22} + K_{24}) \big(\delta^p_i \delta^q_k - \delta^q_i \delta^p_k \big) \big) \left(\partial_i D^k_q \right)^+ - D^k_q \big(K_{11} \delta^p_i \delta^q_k + K_{22} e^p_i e^q_k d\, d + K_{33} \delta_{ik} \delta^p \delta^q d\, d \\
& - (K_{22} + K_{24}) \big(\delta^p_i \delta^q_k - \delta^q_i \delta^p_k \big) \big) \frac{n_i}{u} \left(D^j_p \right) - D^j_p \big(K_{11} \delta^p_i \delta^q_k + K_{22} e^p_{mi} e^q_{lk} d^m d^l + K_{33} \delta_{ik} \delta^p_l \delta^q_m d^m d^l \\
& - (K_{22} + K_{24}) \big(\delta^p_i \delta^q_k - \delta^q_i \delta^p_k \big) \big) \frac{n_i}{u} \left(D^k_q \right), \quad i = 2, 3, 4,
\end{aligned}$$

$$\mathscr{B}_{i11} = -\left(2\rho\partial_\rho Q + \partial^2_{\lambda\rho}Q\lambda\right)\frac{n_i}{u}, \quad i = 2,3,4,$$

$$\mathscr{C}_{i8} = \left(-2\rho\dot{\rho}\partial_\rho Q + 2\rho^2\partial^2_{\rho\vartheta}Q + \dot{\rho}\lambda\partial^2_{\lambda\dot{\vartheta}}Q + \rho\lambda\partial^2_{\lambda\vartheta}Q + \rho\dot{\lambda}\partial^2_{\lambda\dot{\vartheta}}Q\right)\frac{n_i}{u} - \left(2\rho\partial^2_{\rho\dot{\vartheta}}Q - \lambda\partial^2_{\lambda\dot{\vartheta}}Q\right)(\partial_i\rho)^+, \quad i = 2,3,4,$$

$$\mathscr{B}_{i18} = -\frac{n_i}{u}\left(2\rho\partial^2_{\rho\dot{\vartheta}}Q + \lambda\partial^2_{\lambda\dot{\vartheta}}Q\right), \quad i = 2,3,4,$$

$$\mathscr{C}_{ij} = D^k_q\left(K_{11}\delta^p_j\delta^q_k + K_{22}e^p_{mj}e^q_{lk}d^m d^l + K_{33}\delta_{jk}\delta^p_l\delta^q_m d^m d^l - (K_{22}+K_{24})\left(\delta^p_j\delta^q_k - \delta^\gamma_j\delta^q_k\right)\right)\frac{n_i n_p}{u^2}(\dot{\rho})^+ - D^k_q\left(K_{11}\delta^p_j\delta^q_k + K_{22}e^p_{mj}e^q_{lk}d^m d^l + K_{33}\delta_{jk}\delta^p_l\delta^q_m d^m d^l - D^k_q(K_{22}+K_{24})\left(\delta^p_j\delta^q_k - \delta^q_j\delta^p_k\right)\right)\frac{n_p}{u}(\dot{\rho})^+ + K_{33}\frac{\delta_{li}}{u}\left(n_k d^k D^l_j + n_j d^k D^l_k\right), \quad i = 2,3,4;\ j = 5,6,7,$$

$$\mathscr{B}_{ijk} = 2\rho\left\{K_{11}\delta^p_j\delta^q_k + K_{22}e^p_{mj}e^q_{kl}d^m d^k + K_{33}\delta_{jk}\delta^p_l\delta^q_m d^m d^l - (K_{22}+K_{24})\left(\delta^p_j\delta^q_k - \delta^q_j\delta^p_k\right)\right\}\frac{n_i n_p n_q}{u^3}, \quad i = 2,3,4;\ j,k = 5,6,7,$$

$$\mathscr{B}_{i1k} = -2D^j_p\left\{\left(K_{11}\delta^p_j\delta^q_k + K_{22}e^p_{mj}e^q_{kl}d^m d^k + K_{33}\delta_{jk}\delta^p_l\delta^q_m d^m d^l - (K_{22}+K_{24})\left(\delta^p_j\delta^q_k - \delta^q_j\delta^p_k\right)\right)\frac{n_i n_q}{u^2} + \left(K_{11}\delta^p_j\delta^q_k D^k_q + K_{22}e^p_{mj}e^q_{kl}d^m d^k D^l_q + K_{33}\delta_{jk}\delta^p_l\delta^q_m d^m d^l D^k_q - D^k_q(K_{22}+K_{24})\left(\delta^p_j\delta^q_k - \delta^q_j\delta^p_k\right)\right)\frac{n_i n_p}{u^2}\right\}, \quad i,k = 5,6,7,$$

$$\mathscr{C}_{i3} = 2\left(K_{22}e^p_{ki}e^q_{lj} + K_{33}\delta_{ij}\delta^p_k\delta^q_l\right)d^k D^l_q\left(\dot{D}^j_p\right)^+ - \frac{n_p}{u}\left\{(K_{11} - K_{22} + K_{24})\delta^p_i\delta^q_j + (K_{22}+K_{24})\delta^p_j\delta^q_i + \left(K_{22}e^p_{ki}e^q_{lj} + K_{33}\delta_{ij}\delta^p_k\delta^q_l\right)d^k d^l\right\}\left(\dot{D}^j_q\right)^+, \quad i = 5,6,7,$$

$$\mathscr{C}_{ij} = \left\{(K_{11} + 2K_{24})\delta^p_i\delta^q_j + \left(K_{22}e^p_{ki}e^q_{lj} + K_{33}\delta_{ij}\delta^p_k\delta^q_l\right)d^k d^l\right\}\frac{n_p n_q}{u^2} + 2\left(K_{22}e^p_{ji}e^q_{lk} + K_{33}\delta_{ik}\delta^p_j\delta^q_l\right)d^l\dot{d}^k\frac{n_p n_q}{u^2} - 2\rho\left(K_{22}e^p_{ki}e^q_{lj} + K_{33}\delta_{ij}\delta^p_k\delta^q_l\right)d^k D^j_q\frac{n_p}{u} - 2\rho\left(K_{22}e^p_{ki}e^q_{lj} + K_{33}\delta_{ij}\delta^p_k\delta^q_l\right)d^l D^k_p\frac{n_q}{u}, \quad i,j = 5,6,7,$$

$$\mathscr{B}_{i1k} = -\left\{(K_{11} + K_{24})\,\delta^p_i\delta^q_k + \left(K_{22}e^p_{ki}e^q_{lj} + K_{33}\delta_{ik}\delta^p_j\delta^q_l\right)d^j d^l\right\}\frac{n_p n_q}{u^2}, \quad i,j = 5,6,7,$$

$$\mathscr{C}_{81} = -\partial^2_{\rho\rho}K^{pq}\partial_q\vartheta\left((\partial_p\rho)^+ - \frac{n_p}{u}(\dot{\rho})^+\right) - \partial^2_{\rho\vartheta}K^{pq}\left(\partial_p\vartheta\right)\left(\partial_q\vartheta\right) - \partial^2_{\rho\dot{\vartheta}}K^{pq}\left(\partial_p\vartheta\right)(\partial_q\dot{\vartheta}) - \partial^2_{\rho\lambda}K^{pq}\left(\partial_p\vartheta\right)\left(\partial_q\lambda\right) + \partial^2_{\rho\dot{\vartheta}}K^{pq}\frac{n_p}{u}\left(\partial_q\vartheta\right)(\ddot{\vartheta})^+\partial^2_{\rho\lambda}K^{pq}\left(\partial_q\vartheta\right)\dot{\lambda}\frac{n_p}{u} - \vartheta(\dot{\eta})^+(\vartheta(\dot{\rho})^+\rho\dot{\vartheta})\partial^2_{\rho\vartheta}Q,$$

$$\mathscr{C}_{88} = \partial^2_{\rho\dot\vartheta}K^{ij}\partial_i\vartheta\left(\partial_j\rho\right)^+ - \partial^2_{\vartheta\dot\vartheta}K^{ij}(\partial_i\vartheta)\left(\partial_j\vartheta\right) - \partial^2_{\dot\vartheta\dot\vartheta}K^{ij}\left(\partial_j\vartheta\right)\left(\partial_i\dot\vartheta\right)^+ + \partial^2_{\dot\vartheta\dot\vartheta}K^{ij}(\partial_i\vartheta)\frac{n_j}{u}(\ddot\vartheta)^+$$
$$+ \partial^2_{\lambda\dot\vartheta}K^{ij}\left(\partial_j\vartheta\right)(\partial_i\lambda) + \partial^2_{\dot\lambda\dot\vartheta}K^{ij}\left(\partial_j\vartheta\right)\frac{n_i}{u}\dot\lambda + (\vartheta(\dot\rho)^+\rho\dot\vartheta)\partial^2_{\vartheta\dot\vartheta}Q,$$

$$\mathscr{C}_{8i} = -\frac{n_p n_q}{u^2}\partial^2_{d^j D^i_q}K^{pr}\partial_r\vartheta\dot d^j + \frac{n_q}{u}\partial^2_{d^j D^i_q}K^{pr}\partial_r\vartheta D^j_p - \frac{n_p n_q}{u^2}\partial_{D^i_q}K^{pr}\partial_r\vartheta$$
$$- \partial^2_{D^i_s D^k_q}K^{pr}\partial_q\vartheta\left(-\frac{n_s}{u}\left(\partial_p D^k_s\right)^+ + \frac{n_p n_s}{u^2}\left(\dot D^k_s\right)^+\right), \quad i = 5, 6, 7,$$

$$\mathscr{B}_{811} = -\partial^2_{\rho\rho}K^{ij}\partial_j\vartheta\frac{n_i}{u} + \vartheta\partial^2_{\rho\vartheta}Q,$$

$$\mathscr{B}_{888} = -\partial^2_{\dot\vartheta\dot\vartheta}K^{ij}\partial_j\vartheta\frac{n_i}{u},$$

$$\mathscr{B}_{818} = K^{ij}\partial_j\vartheta\frac{n_i}{u} + \partial^2_{\rho\dot\vartheta}K^{ij}\partial_j\vartheta\frac{n_i}{u} - \vartheta\partial^2_{\vartheta\dot\vartheta}Q,$$

$$\mathscr{B}_{8jk} = -\partial^2_{D^j_s D^k_q}K^{pr}\partial_r\vartheta\frac{n_p n_q n_s}{u^3}, \quad j,k = 5, 6, 7.$$

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